The Physical Quantities Corresponding to the Indices

The TKNN Formula for the Quantum Hall Conductance

Our goal in this section is to show the quantum Hall conductance of a bulk system.

The first thing we do is analyze the linear response formula of Kubo (perturbation theory in quantum mechanics). This will give us the expectation value of an observable in the presence of a perturbation.

- Write $H = H_0 + V(t)$. V is the perturbation. It is written now with time dependence, but ultimately we will assume that long ago in the past it is zero and in the far future it is constant.
- $\rho(t) = \rho_0 + \Delta\rho(t)$ where ρ_0 is a stationary-state of H_0 (that means $\dot{\rho_0} = [\rho_0, H_0] = 0$)
- ρ obeys the Liouville equation $\dot{\rho} = \frac{1}{i\hbar}[H, \rho]$.
- Switch to the interaction picture:
  - $\rho^I(t) := e^{-\frac{1}{i\hbar}H_0 t}\rho e^{\frac{1}{i\hbar}H_0 t}$
  - ${\rho_0}^I = e^{-\frac{1}{i\hbar}H_0 t}\rho_0 e^{\frac{1}{i\hbar}H_0 t} = \rho_0$
  - $\Delta\rho^I(t) = e^{-\frac{1}{i\hbar}H_0 t}\Delta\rho(t)\, e^{\frac{1}{i\hbar}H_0 t}$
  - ${H_0}^I = e^{-\frac{1}{i\hbar}H_0 t}H_0 e^{\frac{1}{i\hbar}H_0 t} = H_0$

1.1. CLAIM. $i\hbar\frac{d}{dt}\Delta\rho^I = \left[V^I, \rho_0\right]$.

PROOF.

- Reverse the definition to get $\rho = e^{\frac{1}{i\hbar}H_0 t}\rho^I(t)\, e^{-\frac{1}{i\hbar}H_0 t}$.
- Plug this into the Liouville equation to get

$$\begin{aligned}
\frac{d}{dt}\left(e^{\frac{1}{i\hbar}H_0 t}\rho^I(t)\, e^{-\frac{1}{i\hbar}H_0 t}\right) &= \frac{1}{i\hbar}\left[H, e^{\frac{1}{i\hbar}H_0 t}\rho^I(t)\, e^{-\frac{1}{i\hbar}H_0 t}\right] \\
\frac{1}{i\hbar}e^{\frac{1}{i\hbar}H_0 t}H_0\rho^I(t)\, e^{-\frac{1}{i\hbar}H_0 t} &= \frac{1}{i\hbar}\left[H_0 + V, e^{\frac{1}{i\hbar}H_0 t}\rho^I(t)\, e^{-\frac{1}{i\hbar}H_0 t}\right] \\
-\frac{1}{i\hbar}e^{\frac{1}{i\hbar}H_0 t}\rho^I(t)\, H_0 e^{-\frac{1}{i\hbar}H_0 t} & \\
+e^{\frac{1}{i\hbar}H_0 t}\left(\frac{d}{dt}\rho^I(t)\right)e^{-\frac{1}{i\hbar}H_0 t} & \\
\frac{1}{i\hbar}\left[H_0, \rho^I(t)\right] + \frac{d}{dt}\rho^I(t) &= \frac{1}{i\hbar}e^{-\frac{1}{i\hbar}H_0 t}\left[H_0 + V, e^{\frac{1}{i\hbar}H_0 t}\rho^I(t)\, e^{-\frac{1}{i\hbar}H_0 t}\right]e^{\frac{1}{i\hbar}H_0 t} \\
\frac{1}{i\hbar}\left[H_0, \rho^I(t)\right] + \frac{d}{dt}\rho^I(t) &= \frac{1}{i\hbar}\left[(H_0)^I + V^I, \rho^I(t)\right] \\
\frac{d}{dt}\rho^I(t) &= \frac{1}{i\hbar}\left[V^I, \rho^I(t)\right]
\end{aligned}$$

- But $\frac{d}{dt}\left(\rho^I(t)\right) = \frac{d}{dt}\left(\rho_0 + \Delta\rho^I\right) = \frac{d}{dt}\Delta\rho^I$, and $\left[V^I, \Delta\rho^I\right] \propto \mathcal{O}\left(V^2\right)$ (because in perturbation theory $\Delta\rho \propto V$)
- Thus we get $\frac{d}{dt}\Delta\rho^I = \frac{1}{i\hbar}\left[V^I, \rho_0\right]$.

□

- Assume that

$$\lim_{t\to-\infty}\Delta\rho(t) = 0$$

1.2. CLAIM. We can integrate the equation of motion to get:

$$\Delta\rho^I = \frac{1}{i\hbar}\int_{-\infty}^{t}\left[V^I(t'), \rho_0\right]dt'$$

PROOF.

- Differentiate this Ansatz to verify its validity.

□

- Let B be some observable.
- Then $\langle B(t)\rangle = \mathrm{Tr}(B\rho(t))$.
  - Due to the cyclicity of the trace we have

$$\begin{aligned}\langle B(t)\rangle &= \mathrm{Tr}\left(Be^{-\frac{i}{\hbar}H_0t}e^{\frac{i}{\hbar}H_0t}\rho(t)\,e^{-\frac{i}{\hbar}H_0t}e^{\frac{i}{\hbar}H_0t}\right)\\ &= \mathrm{Tr}\left(e^{\frac{i}{\hbar}H_0t}Be^{-\frac{i}{\hbar}H_0t}e^{\frac{i}{\hbar}H_0t}\rho(t)\,e^{-\frac{i}{\hbar}H_0t}\right)\\ &= \mathrm{Tr}\left(B^I\rho^I\right)\\ &= \mathrm{Tr}\left(B^I\left((\rho_0)^I+(\Delta\rho)^I\right)\right)\\ &= \mathrm{Tr}\left(B^I(\rho_0)^I\right)+\mathrm{Tr}\left(B^I\Delta\rho^I\right)\end{aligned}$$

- Assume B is such that $\mathrm{Tr}\left(B^I(\rho_0)^I\right)=0$.
- Then

$$\begin{aligned}\langle B(t)\rangle &= \mathrm{Tr}\left(B^I(t)\,\Delta\rho^I(t)\right)\\ &= \mathrm{Tr}\left(B^I(t)\left(\frac{1}{i\hbar}\int_{-\infty}^{t}\left[V^I(t'),\rho_0\right]dt'\right)\right)\\ &= \frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(B^I(t)\left[V^I(t'),\rho_0\right]\right)dt'\\ &= \frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(B^I(t)\,V^I(t')\,\rho_0-B^I(t)\,\rho_0V^I(t')\right)dt'\\ &= \frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0B^I(t)\,V^I(t')-\rho_0V^I(t')\,B^I(t)\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0V^I(t')\,B^I(t)-\rho_0B^I(t)\,V^I(t')\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0\left[V^I(t'),B^I(t)\right]\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0\left[e^{-\frac{1}{i\hbar}H_0t'}V(t')\,e^{\frac{1}{i\hbar}H_0t'},\,e^{-\frac{1}{i\hbar}H_0t}Be^{\frac{1}{i\hbar}H_0t}\right]\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0e^{-\frac{1}{i\hbar}H_0t'}V(t')\,e^{\frac{1}{i\hbar}H_0t'}e^{-\frac{1}{i\hbar}H_0t}Be^{\frac{1}{i\hbar}H_0t}-\rho_0e^{-\frac{1}{i\hbar}H_0t}Be^{\frac{1}{i\hbar}H_0t}e^{-\frac{1}{i\hbar}H_0t'}V(t')\,e^{\frac{1}{i\hbar}H_0t'}\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(e^{-\frac{1}{i\hbar}H_0t'}\rho_0V(t')\,e^{\frac{1}{i\hbar}H_0t'}e^{-\frac{1}{i\hbar}H_0t}Be^{\frac{1}{i\hbar}H_0t}-e^{\frac{1}{i\hbar}H_0t'}\rho_0e^{-\frac{1}{i\hbar}H_0t}Be^{\frac{1}{i\hbar}H_0t}e^{-\frac{1}{i\hbar}H_0t'}V(t')\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0V(t')\,e^{\frac{1}{i\hbar}H_0t'}e^{-\frac{1}{i\hbar}H_0t}Be^{\frac{1}{i\hbar}H_0t}e^{-\frac{1}{i\hbar}H_0t'}-\rho_0e^{\frac{1}{i\hbar}H_0t'}e^{-\frac{1}{i\hbar}H_0t}Be^{\frac{1}{i\hbar}H_0t}e^{-\frac{1}{i\hbar}H_0t'}V(t')\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0V(t')\,e^{-\frac{1}{i\hbar}H_0(t-t')}Be^{\frac{1}{i\hbar}H_0(t-t')}-\rho_0e^{-\frac{1}{i\hbar}H_0(t-t')}Be^{\frac{1}{i\hbar}H_0(t-t')}V(t')\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0V(t')\,B^I(t-t')-\rho_0B^I(t-t')\,V(t')\right)dt'\\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0\left[V(t'),B^I(t-t')\right]\right)dt'\end{aligned}$$

- Make a Fourier transform of V as

$$V(t)=\lim_{\eta\to0}\frac{1}{2\pi}\int_{\mathbb{R}}e^{-i(\omega+i\eta)t}\tilde{V}(\omega)\,d\omega$$

where $\eta>0$ is some factor we introduce to insure the boundary condition $\lim_{t\to-\infty}\Delta\rho(t)=0$ is met.

- Then

$$\begin{aligned}\langle B(t)\rangle &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0\left[V(t'),B^I(t-t')\right]\right)dt' \\ &= -\frac{1}{i\hbar}\int_{-\infty}^{t}\mathrm{Tr}\left(\rho_0\left[\left(\lim_{\eta\to0}\frac{1}{2\pi}\int_{\mathbb{R}}e^{-i(\omega+i\eta)t'}\tilde{V}(\omega)\,d\omega\right),B^I(t-t')\right]\right)dt' \\ &= \lim_{\eta\to0}\frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\infty}^{t}\frac{i}{\hbar}\mathrm{Tr}\left(\rho_0\left[\tilde{V}(\omega),B^I(t-t')\right]\right)e^{-i(\omega+i\eta)t'}dt'd\omega \\ &= \lim_{\eta\to0}\frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\infty}^{\infty}\frac{i}{\hbar}\mathrm{Tr}\left(\rho_0\left[\tilde{V}(\omega),B^I(t-t')\right]\right)\theta(t-t')\,e^{-i(\omega+i\eta)t'}dt'd\omega\end{aligned}$$

1.3. DEFINITION. Define the linear response function

$$\chi_{B\tilde{V}(\omega)}(t):=\frac{i}{\hbar}\mathrm{Tr}\left(\rho_0\left[\tilde{V}(\omega),B^I(t)\right]\right)\theta(t)$$

- Observe we can again change the order to write

$$\begin{aligned}\chi_{B\tilde{V}(\omega)}(t) &\equiv \frac{i}{\hbar}\mathrm{Tr}\left(\rho_0\left[\tilde{V}(\omega),B^I(t)\right]\right)\theta(t) \\ &= \frac{i}{\hbar}\mathrm{Tr}\left(\rho_0\tilde{V}(\omega)B^I(t)-\rho_0B^I(t)\tilde{V}(\omega)\right)\theta(t) \\ &= \frac{i}{\hbar}\mathrm{Tr}\left(\rho_0\tilde{V}(\omega)B^I(t)-\tilde{V}(\omega)\rho_0B^I(t)\right)\theta(t) \\ &= -\frac{i}{\hbar}\mathrm{Tr}\left(\left[\tilde{V}(\omega),\rho_0\right]B^I(t)\right)\theta(t)\end{aligned}$$

Then we find

$$\boxed{\langle B(t)\rangle=\lim_{\eta\to0}\frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\infty}^{\infty}\chi_{B\tilde{V}(\omega)}(t-t')\,e^{-i(\omega+i\eta)t'}dt'd\omega}$$

which is the linear response formula.

The next step is to employ this in order to find the Hall conductivity. We will start this analysis in the many-body picture:

- Define

$$B_\mu:=-eJ_\mu(r):=-e\left(\frac{1}{2}\sum_{j=1}^{N}\left(v_{j\mu}\delta(r-r_j)+\delta(r-r_j)\,v_{j\mu}\right)\right)$$

  where $v_{j\mu}$ is the velocity operator of the jth particle along the μth axis.
- Define the operator

$$X_\mu:=\sum_{j=1}^{N}r_{j\mu}$$

- Define

$$\begin{aligned}V(t) &:= \sum_{j=1}^{N}e\sum_{\mu\in\{x,y,z\}}r_{j\mu}E_\mu f(t) \\ &= \sum_\mu eX_\mu E_\mu f(t)\end{aligned}$$

  where E_μ is the electric field in the system and $f(t)$ is some attenuation function which goes to zero at $t\to-\infty$ and "turns on" adiabatically.
- Then the linear response is

$$\begin{aligned}\chi_{B_\mu eX_\nu}(t) &:= \frac{i}{\hbar}\mathrm{Tr}\left(\rho_0\left[eX_\nu,-eJ_\mu{}^I(r,t)\right]\right)\theta(t) \\ &= -\frac{ie^2}{\hbar}\mathrm{Tr}\left(\rho_0\left[X_\nu,J_\mu{}^I(r,t)\right]\right)\theta(t)\end{aligned}$$

- By Ohm's law, $j_a=\sigma_{ab}E_b$, that is, σ_{ab} is exactly the response of the current j to the perturbation E, so we expect $\sigma_{\mu\nu}=\chi_{B_\mu eX_\nu}(t)$ somehow, as we shall see below.

- Assume $(|n\rangle)_{n\in\mathbb{N}}$ is an orthonormal complete set of eigenstates of H_0 (the many-body Hamiltonian) with eigenvalues $(E_n)_{n\in\mathbb{N}}$, where we assume $E_0 = \min(\{E_n | n \in \mathbb{N}\})$. In this basis, ρ_0 is diagonal:

$$\begin{aligned}\rho_0 &= \sum_{n\in\mathbb{N}} |n\rangle\langle n|\,\rho_0 \sum_{m\in\mathbb{N}} |m\rangle\langle m| \\ &= \sum_{(n,\,m)\in\mathbb{N}^2} \langle n\,|\,\rho_0\,|\,m\rangle\, |n\rangle\langle m|\end{aligned}$$

and then $[\rho_0,\, H_0] = 0$ means

$$\begin{aligned}\langle m'|\,[\rho_0,\,H_0]\,|n'\rangle &= \langle m'| \left[\sum_{(n,\,m)\in\mathbb{N}^2} \langle n\,|\,\rho_0\,|\,m\rangle\,|n\rangle\langle m|\,,\, H_0\right] |n'\rangle \\ &= \langle m'| \left(\sum_{(n,\,m)\in\mathbb{N}^2} \langle n\,|\,\rho_0\,|\,m\rangle\,|n\rangle\langle m|\,H_0 - \sum_{(n,\,m)\in\mathbb{N}^2} H_0\,\langle n\,|\,\rho_0\,|\,m\rangle\,|n\rangle\langle m|\right) |n'\rangle \\ &= \langle m'| \left(\sum_{(n,\,m)\in\mathbb{N}^2} \langle n\,|\,\rho_0\,|\,m\rangle\,|n\rangle\langle m|\,E_m - \sum_{(n,\,m)\in\mathbb{N}^2} E_n\,\langle n\,|\,\rho_0\,|\,m\rangle\,|n\rangle\langle m|\right) |n'\rangle \\ &= \langle m'| \sum_{(n,\,m)\in\mathbb{N}^2} \langle n\,|\,\rho_0\,|\,m\rangle\,|n\rangle\langle m|\,(E_m - E_n)\,|n'\rangle \\ &= \langle m'\,|\,\rho_0\,|\,n'\rangle\,(E_{n'} - E_{m'}) \\ &\overset{!}{=} 0\end{aligned}$$

and so if $(E_{n'} - E_{m'}) \neq 0$ then $\langle m'\,|\,\rho_0\,|\,n'\rangle = 0$. Assuming we don't have degeneracy, this means $\langle m'\,|\,\rho_0\,|\,n'\rangle \propto \delta_{m'n'}$.

- Then

$$\begin{aligned}\mathrm{Tr}\left(\rho_0\left[X_\nu,\,J_\mu{}^I(r,\,t)\right]\right) &= \sum_{n\in\mathbb{N}} \left\langle n\,\middle|\,\rho_0\left[X_\nu,\,J_\mu{}^I(r,\,t)\right]\middle|\,n\right\rangle \\ &= \sum_{n\in\mathbb{N}} \left\langle n\,\middle|\,\sum_{m\in\mathbb{N}} \langle m\,|\,\rho_0\,|\,m\rangle\,|m\rangle\langle m|\left[X_\nu,\,J_\mu{}^I(r,\,t)\right]\middle|\,n\right\rangle \\ &= \sum_{n\in\mathbb{N}} \langle n\,|\,\rho_0\,|\,n\rangle \left\langle n\,\middle|\left[X_\nu,\,J_\mu{}^I(r,\,t)\right]\middle|\,n\right\rangle\end{aligned}$$

  - In the canonical ensemble,

$$\begin{aligned}\langle n|\rho_0|n\rangle &= \left\langle n\,\middle|\left(\frac{e^{-\beta H_0}}{\mathrm{Tr}\left(e^{-\beta H_0}\right)}\right)\middle|\,n\right\rangle \\ &= \left\langle n\,\middle|\left(\frac{e^{-\beta E_n}}{Z}\right)\middle|\,n\right\rangle \\ &= \frac{e^{-\beta E_n}}{Z}\end{aligned}$$

  - When $T \to 0$, we can make the approximation $\langle n|\rho_0|n\rangle \approx \delta_{n0}$. We also assume that in the ground state there is no current, that is, $\langle 0\,|\,J_\mu(r)\,|\,0\rangle = 0$.

- Thus we have

$$\begin{aligned}
\mathrm{Tr}\left(\rho_0\left[X_\nu, J_\mu{}^I(r,t)\right]\right) &= \sum_{n\in\mathbb{N}} \langle n|\rho_0|n\rangle \left\langle n\middle|\left[X_\nu, J_\mu{}^I(r,t)\right]\middle|n\right\rangle \\
&\approx \left\langle 0\middle|\left[X_\nu, J_\mu{}^I(r,t)\right]\middle|0\right\rangle \\
&= \left\langle 0\middle|X_\nu J_\mu{}^I(r,t)\middle|0\right\rangle - \left\langle 0\middle|J_\mu{}^I(r,t)X_\nu\middle|0\right\rangle \\
&= \left\langle 0\middle|X_\nu \sum_{n\in\mathbb{N}}|n\rangle\langle n|J_\mu{}^I(r,t)\middle|0\right\rangle - \left\langle 0\middle|J_\mu{}^I(r,t)\sum_{n\in\mathbb{N}}|n\rangle\langle n|X_\nu\middle|0\right\rangle \\
&= \sum_{n\in\mathbb{N}} \langle 0|X_\nu|n\rangle \left\langle n\middle|J_\mu{}^I(r,t)\middle|0\right\rangle - \left\langle 0\middle|J_\mu{}^I(r,t)\middle|n\right\rangle\langle n|X_\nu|0\rangle \\
&= \sum_{n\in\mathbb{N}} \langle 0|X_\nu|n\rangle \left\langle n\middle|e^{-\frac{1}{i\hbar}H_0 t}J_\mu(r)e^{\frac{1}{i\hbar}H_0 t}\middle|0\right\rangle - \left\langle 0\middle|e^{-\frac{1}{i\hbar}H_0 t}J_\mu(r)e^{\frac{1}{i\hbar}H_0 t}\middle|n\right\rangle\langle n|X_\nu|0\rangle \\
&= \sum_{n\in\mathbb{N}} \langle 0|X_\nu|n\rangle \left\langle n\middle|e^{-\frac{1}{i\hbar}E_n t}J_\mu(r)e^{\frac{1}{i\hbar}E_0 t}\middle|0\right\rangle - \left\langle 0\middle|e^{-\frac{1}{i\hbar}E_0 t}J_\mu(r)e^{\frac{1}{i\hbar}E_n t}\middle|n\right\rangle\langle n|X_\nu|0\rangle \\
&= \sum_{n\in\mathbb{N}} \langle 0|X_\nu|n\rangle\langle n|J_\mu(r)|0\rangle e^{-\frac{1}{i\hbar}(E_n-E_0)t} - \langle 0|J_\mu(r)|n\rangle\langle n|X_\nu|0\rangle e^{\frac{1}{i\hbar}(E_n-E_0)t}
\end{aligned}$$

- So we find that

$$\lim_{T\to 0}\chi_{B_\mu e X_\nu}(t) \approx -\frac{ie^2}{\hbar}\theta(t)\sum_{n\in\mathbb{N}}\left(\langle 0|X_\nu|n\rangle\langle n|J_\mu(r)|0\rangle e^{-\frac{1}{i\hbar}(E_n-E_0)t} - \langle 0|J_\mu(r)|n\rangle\langle n|X_\nu|0\rangle e^{\frac{1}{i\hbar}(E_n-E_0)t}\right)$$

- Thus the expectation value for the electric current is given by:

$$\begin{aligned}
\lim_{T\to 0}\langle B_\mu(t)\rangle &= \sum_\nu \int_{-\infty}^{\infty} \lim_{T\to 0}\chi_{B_\mu e X_\nu}(t-t')E_\nu f(t')\,dt' \\
&= \sum_\nu \underbrace{\int_{-\infty}^{\infty} \lim_{T\to 0}\chi_{B_\mu e X_\nu}(t-t') f(t')\,dt'}_{\sigma_{\mu\nu}} E_\nu
\end{aligned}$$

- And as such we found a formula:

$$\begin{aligned}
\lim_{T\to\infty}\sigma_{\mu\nu}(r,t) &= \int_{-\infty}^{\infty}\lim_{T\to 0}\chi_{B_\mu e X_\nu}(t-t')f(t')\,dt'd\omega \\
&\approx -\frac{ie^2}{\hbar}\int_{-\infty}^{\infty}\theta(t-t')\sum_{n\in\mathbb{N}}\left(\langle 0|X_\nu|n\rangle\langle n|J_\mu(r)|0\rangle e^{-\frac{1}{i\hbar}(E_n-E_0)(t-t')}\right)f(t')\,dt' \\
&\quad -\frac{ie^2}{\hbar}\int_{-\infty}^{\infty}\theta(t-t')\sum_{n\in\mathbb{N}}\left(-\langle 0|J_\mu(r)|n\rangle\langle n|X_\nu|0\rangle e^{\frac{1}{i\hbar}(E_n-E_0)(t-t')}\right)f(t')\,dt'
\end{aligned}$$

- The actual conductivity is time independent and so we need to integrate this over time:

$$\sigma_{\mu\nu}(r) = \lim_{\eta\to 0^+}\int_{-\infty}^{\infty} dt e^{-\eta t}\sigma_{\mu\nu}(r,t)$$

  where $e^{-\eta t}$ is a convergence factor with $\eta > 0$.
- Denote by ν_μ the total velocity, that is, $\nu_\mu = \int d^2 r J_\mu(r)$.
- We are interested in the conductivity of the material as a whole, and not just at one particular point, so we should average over space. Thus, if A is the area of the material, the final quantity we are interested

in is

$$
\begin{aligned}
\sigma_{\mu\nu} &= \frac{1}{A}\int d^2r \lim_{\eta\to 0^+}\int_{-\infty}^{\infty} dt e^{-\eta t}\sigma_{\mu\nu}(r,t) \\
&= \frac{1}{A}\int d^2r \lim_{\eta\to 0^+}\int_{-\infty}^{\infty} dt e^{-\eta t}\int_{-\infty}^{\infty} -\frac{ie^2}{\hbar}\theta\left(t-t'\right)\times \\
&\times \sum_{n\in\mathbb{N}}\left(\langle 0|X_\nu|n\rangle\langle n|J_\mu(r)|0\rangle e^{-\frac{1}{i\hbar}(E_n-E_0)(t-t')} - \langle 0|J_\mu(r)|n\rangle\langle n|X_\nu|0\rangle e^{\frac{1}{i\hbar}(E_n-E_0)(t-t')}\right) f\left(t'\right) dt' \\
&= e^2\sum_{n\in\mathbb{N}}\left(-\frac{i}{\hbar}\frac{1}{A}\right)\lim_{\eta\to 0^+}\int_{-\infty}^{\infty} dt e^{-\eta t}\int_{-\infty}^{\infty}\theta\left(t-t'\right)\times \\
&\times\left(\langle 0|X_\nu|n\rangle\langle n|v_\mu|0\rangle e^{-\frac{1}{i\hbar}(E_n-E_0)(t-t')} - \langle 0|v_\mu|n\rangle\langle n|X_\nu|0\rangle e^{\frac{1}{i\hbar}(E_n-E_0)(t-t')}\right) f\left(t'\right) dt' \\
&= \frac{e^2}{A}\sum_{n\in\mathbb{N}}\frac{\langle 0|X_\nu|n\rangle\langle n|v_\mu|0\rangle + \langle 0|v_\mu|n\rangle\langle n|X_\nu|0\rangle}{E_n-E_0}
\end{aligned}
$$

- The last step is to note that the velocity operator is equal to $v_\mu = \dot{X}_\mu$, and of course by the Heisenberg equation of motion we then have $v_\mu = \frac{1}{i\hbar}[X_\mu, H_0]$ or $X_\mu H_0 - H_0 X_0 = i\hbar v_\mu$. As a result,

$$
\begin{aligned}
\langle 0|X_\nu|n\rangle &= \frac{E_n-E_0}{E_n-E_0}\langle 0|X_\nu|n\rangle \\
&= \frac{1}{E_n-E_0}\langle 0|(E_n-E_0)X_\nu|n\rangle \\
&= \frac{1}{E_n-E_0}\langle 0|(X_\nu E_n - E_0 X_\nu)|n\rangle \\
&= \frac{1}{E_n-E_0}\langle 0|(X_\nu H_0 - H_0 X_\nu)|n\rangle \\
&= \frac{1}{E_n-E_0}\langle 0|[X_\nu, H_0]|n\rangle \\
&= \frac{1}{E_n-E_0}\langle 0|i\hbar v_\nu|n\rangle \\
&= \frac{i\hbar}{E_n-E_0}\langle 0|v_\nu|n\rangle
\end{aligned}
$$

- Thus we can write $\sigma_{\mu\nu}$ with only v's as:

$$
\sigma_{\mu\nu} = \frac{ie^2\hbar}{A}\sum_{n\in\mathbb{N}}\frac{\langle 0|v_\nu|n\rangle\langle n|v_\mu|0\rangle - \langle 0|v_\mu|n\rangle\langle n|v_\nu|0\rangle}{(E_n-E_0)^2}
$$

which is the many body equivalent of the sum on occupied single-particle states
to show this we have to use the formula $v = \frac{1}{\hbar}\frac{\partial E(k)}{\partial k}$

2. The Edge Quantum Hall Conductance

The goal in this section is to show that 6.1 actually is equal to the quantum Hall condutance of an edge system. We assume the chemical potential on one edge is μ_+ and μ_- on the other edge, where $\mu_+ \neq \mu_-$ (otherwise the current on one edge cancels out the current on the other edge as they flow in opposite directions). Using the formula

$j=\rho\nu$ where ρ is the density of carriers and ν is the velocity of the carriers, we have

$$\begin{aligned}
I &= \frac{1}{2\pi}\sum_j \int_{k_-^j}^{k_+^j} \nu\left(k\right) dk \\
&= \frac{1}{2\pi}\sum_j \int_{k_-^j}^{k_+^j} \frac{1}{\hbar}\frac{\partial E}{\partial k} dk \\
&= \frac{1}{\hbar}\frac{1}{2\pi}\sum_j \left[E\left(k_+^j\right) - E\left(k_-^j\right)\right] \\
&= \frac{1}{\hbar}\frac{1}{2\pi}\sum_j \left[\mu_+ - \mu_-\right] \\
&= \frac{1}{\hbar}\frac{1}{2\pi}\sum_j V
\end{aligned}$$

where the sum is on intersection points of either μ_+ or μ_- with the gapless edge states, ν is the velocity, and V is the potential between the two edges. Thus we obtain that for each ascending crossing of the gapless edge mode with either μ_+ or μ_- we must count $+1$ for the conductance (given by $\sigma=\frac{I}{V}$) and -1 for a descending crossing.